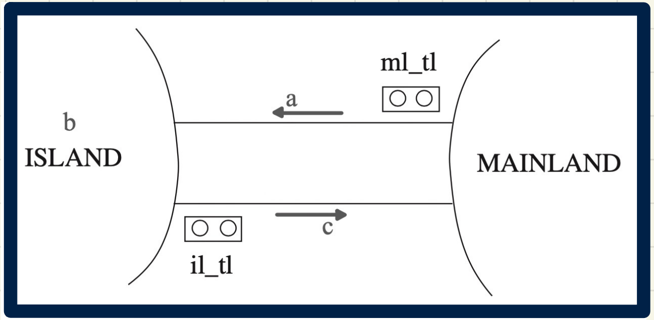
Fixing m2: Adding Actions



ML_tl_green
when
$ml\_tl = red$
$a + b < d$
$c = 0$
then
$ml\_tl := green$
$il\_tl := red$
end

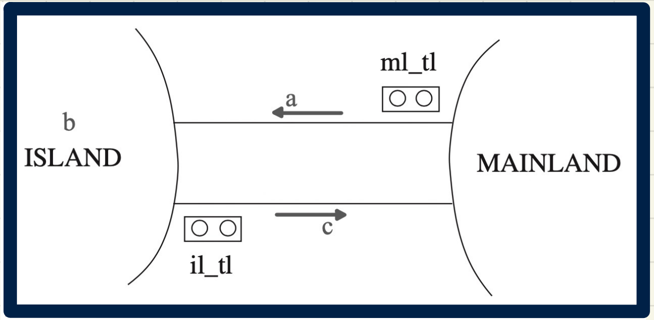
IL_tl_green
when
$il\_tl = red$
$b > 0$
$a = 0$
then
$il\_tl := green$
$ml\_tl := red$
end

ML_tl_green/inv2_5/INV

axm0_1	$d \in \mathbb{N}$
axm0_2	$d > 0$
axm2_1	$COLOUR = \{green, red\}$
axm2_2	$green \neq red$
inv0_1	$n \in \mathbb{N}$
inv0_2	$n \leq d$
inv1_1	$a \in \mathbb{N}$
inv1_2	$b \in \mathbb{N}$
inv1_3	$c \in \mathbb{N}$
inv1_4	$a + b + c = n$
inv1_5	$a = 0 \lor c = 0$
inv2_1	$ml\_tl \in COLOUR$
inv2_2	$il\_tl \in COLOUR$
inv2_3	$ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$il\_tl = green \Rightarrow b > 0 \land a = 0$
inv2_5	$ml\_tl = red \lor il\_tl = red$

Exercise: Specify IL_tl_green/inv2_5/INV

Invariant Preservation: ML_out/inv2_3/INV



variables:
a, b, c
ml_tl
il_tl

ML_out
when
$ml\_tl = green$
then
$a := a + 1$
end

IL_out
when
$il\_tl = green$
then
$b := b - 1$
$c := c + 1$
end

invariants:
inv2_1 : $ml\_tl \in COLOUR$
inv2_2 : $il\_tl \in COLOUR$
inv2_3 : $ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
inv2_4 : $il\_tl = green \Rightarrow b > 0 \wedge a = 0$

ML_out/inv2_3/INV

axm0_1 $\{\ d \in \mathbb{N}$
axm0_2 $\{\ d > 0$
axm2_1 $\{\ COLOUR = \{green, red\}$
axm2_2 $\{\ green \neq red$
inv0_1 $\{\ n \in \mathbb{N}$
inv0_2 $\{\ n \leq d$
inv1_1 $\{\ a \in \mathbb{N}$
inv1_2 $\{\ b \in \mathbb{N}$
inv1_3 $\{\ c \in \mathbb{N}$
inv1_4 $\{\ a + b + c = n$
inv1_5 $\{\ a = 0 \vee c = 0$
inv2_1 $\{\ ml\_tl \in COLOUR$
inv2_2 $\{\ il\_tl \in COLOUR$
inv2_3 $\{\ ml\_tl = green \Rightarrow a + b < d \wedge c = 0$
inv2_4 $\{\ il\_tl = green \Rightarrow b > 0 \wedge a = 0$
inv2_5 $\{\ ml\_tl = red \vee il\_tl = red$
Concrete guards of *ML_out* $\{\ ml\_tl = green$

$\vdash$

Concrete invariant **inv2_3** with *ML_out*'s effect in the post-state $\{\ ml\_tl = green \Rightarrow (a + 1) + b < d \wedge c = 0$

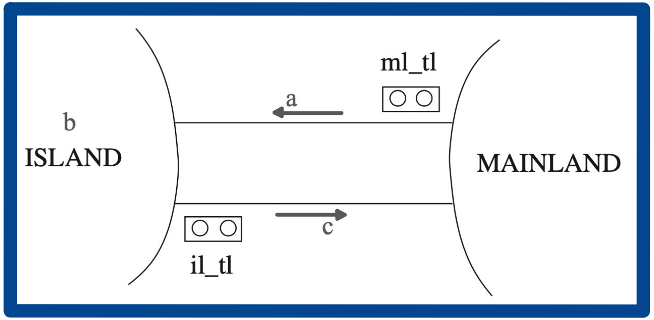
Exercise: Specify IL_out/inv2_4/INV

Discharging POs of m2: Invariant Preservation

First Attempt

$d \in \mathbb{N}$
$d > 0$
$COLOUR = \{green, red\}$
$green \neq red$
$n \in \mathbb{N}$
$n \leq d$
$a \in \mathbb{N}$
$b \in \mathbb{N}$
$c \in \mathbb{N}$
$a + b + c = n$
$a = 0 \lor c = 0$
$ml\_tl \in COLOUR$
$il\_tl \in COLOUR$
$ml\_tl = green \Rightarrow a + b < d \land c = 0$
$il\_tl = green \Rightarrow b > 0 \land a = 0$
$ml\_tl = red \lor il\_tl = red$
$ml\_tl = green$
$\vdash$
$ml\_tl = green \Rightarrow (a+1) + b < d \land c = 0$

ML_out/inv2_3/INV



$$\frac{H \vdash P \qquad H \vdash Q}{H \vdash P \land Q} \quad \textbf{AND\_R}$$

$$\frac{H, P, Q \vdash R}{H, P \land Q \vdash R} \quad \textbf{AND\_L}$$

$$\frac{H, P \vdash Q}{H \vdash P \Rightarrow Q} \quad \textbf{IMP\_R}$$

MON

$ml\_tl = green \Rightarrow a + b < d \land c = 0 \vdash ml\_tl = green \Rightarrow (a+1) + b < d \land c = 0$

IMP_R

$ml\_tl = green \Rightarrow a + b < d \land c = 0,\ ml\_tl = green \vdash (a+1) + b < d \land c = 0$

IMP_R

$a + b < d \land c = 0,\ ml\_tl = green \vdash (a+1) + b < d \land c = 0$

AND_L

$a + b < d,\ c = 0,\ ml\_tl = green \vdash (a+1) + b < d \land c = 0$

AND_R

- $a + b < d,\ c = 0,\ ml\_tl = green \vdash (a+1) + b < d$ **??**
- $a + b < d,\ c = 0,\ ml\_tl = green \vdash c = 0$ **HYP**

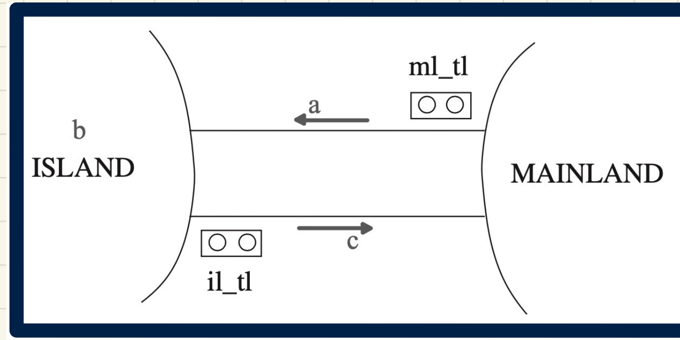
Understanding the Failed Proof on INV

variables:
a, b, c
ml_tl
il_tl

invariants:
inv2_1 : $ml\_tl \in COLOUR$
inv2_2 : $il\_tl \in COLOUR$
inv2_3 : $ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4 : $il\_tl = green \Rightarrow b > 0 \land a = 0$

ML_out
when
$ml\_tl = green$
then
$a := a + 1$
end

IL_out
when
$il\_tl = green$
then
$b := b - 1$
$c := c + 1$
end

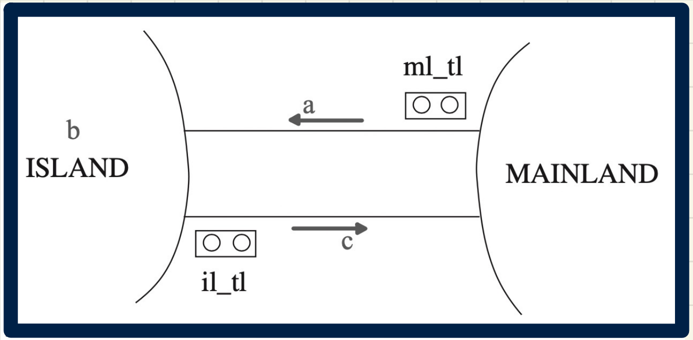


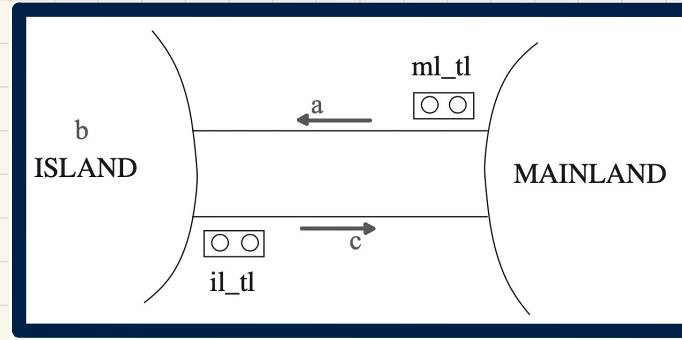
Unprovable Sequent:

$$
\begin{array}{ll}
 & a + b < d \\
\land & c = 0 \\
\land & ml\_tl = green \\
\vdash & \\
 & (a + 1) + b < d
\end{array}
$$

$d = 3, b = 0, a = 0$	[$(a + 1) + b < d$ evaluates to ***true***]
$d = 3, b = 1, a = 0$	[$(a + 1) + b < d$ evaluates to ***true***]
$d = 3, b = 0, a = 1$	[$(a + 1) + b < d$ evaluates to ***true***]
$d = 3, b = 0, a = 2$	[$(a + 1) + b < d$ evaluates to ***false***]
$d = 3, b = 1, a = 1$	[$(a + 1) + b < d$ evaluates to ***false***]
$d = 3, b = 2, a = 0$	[$(a + 1) + b < d$ evaluates to ***false***]

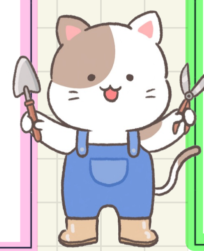
Fixing m2: Splitting Events






```
ML_out_1
  when
    ml_tl = green
    a + b + 1 ≠ d
  then
    a := a + 1
  end
```

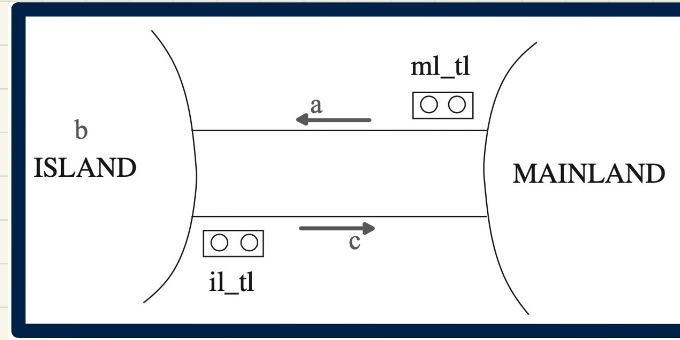
```
ML_out_2
  when
    ml_tl = green
    a + b + 1 = d
  then
    a := a + 1
    ml_tl := red
  end
```



```
IL_out_1
  when
    il_tl = green
    b ≠ 1
  then
    b := b − 1
    c := c + 1
  end
```

```
IL_out_2
  when
    il_tl = green
    b = 1
  then
    b := b − 1
    c := c + 1
    il_tl := red
  end
```

Current m2 May Livelock



ML_tl_green
when
$ml\_tl = red$
$a + b < d$
$c = 0$
then
$ml\_tl := green$
$il\_tl := red$
end

IL_tl_green
when
$il\_tl = red$
$b > 0$
$a = 0$
then
$il\_tl := green$
$ml\_tl := red$
end

$\langle$ *init*, *ML_tl_green*, *ML_out_1*, *IL_in*, *IL_tl_green*, *ML_tl_green*, *IL_tl_green*, $\ldots\rangle$

init	*ML_tl_green*	*ML_out_1*	*IL_in*	*IL_tl_green*	*ML_tl_green*	*IL_tl_green*
$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$	$d = 2$
$a' = 0$	$a' = 0$	$a' = 1$	$a' = 0$	$a' = 0$	$a' = 0$	$a' = 0$
$b' = 0$	$b' = 0$	$b' = 0$	$b' = 1$	$b' = 1$	$b' = 1$	$b' = 1$
$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$	$c' = 0$
$ml\_tl = red$	$ml\_tl' = green$	$ml\_tl' = green$	$ml\_tl' = green$	$ml\_tl' = red$	$ml\_tl' = green$	$ml\_tl' = red$
$il\_tl = red$	$il\_tl' = red$	$il\_tl' = red$	$il\_tl' = red$	$il\_tl' = green$	$il\_tl' = red$	$il\_tl' = green$

Fixing m2: Regulating Traffic Light Changes

Divergence Trace: <init, ML_tl_green, ML_out_1, IL_in, IL_tl_green, ML_tl_green, IL_tl_green, ...>

ML_tl_green
when
$ml\_tl = red$
$a + b < d$
$c = 0$
$il\_pass = 1$
then
$ml\_tl := green$
$il\_tl := red$
$ml\_pass := 0$
end

IL_tl_green
when
$il\_tl = red$
$b > 0$
$a = 0$
$ml\_pass = 1$
then
$il\_tl := green$
$ml\_tl := red$
$il\_pass := 0$
end

ML_out_1
when
$ml\_tl = green$
$a + b + 1 \neq d$
then
$a := a + 1$
$ml\_pass := 1$
end

ML_out_2
when
$ml\_tl = green$
$a + b + 1 = d$
then
$a := a + 1$
$ml\_tl := red$
$ml\_pass := 1$
end

IL_out_1
when
$il\_tl = green$
$b \neq 1$
then
$b := b - 1$
$c := c + 1$
$il\_pass := 1$
end

IL_out_2
when
$il\_tl = green$
$b = 1$
then
$b := b - 1$
$c := c + 1$
$il\_tl := red$
$il\_pass := 1$
end

d = 2	ml_pass	il_pass
< init,	1	1
ML_tl_green,		
ML_out_1,		
ML_out_2,		
IL_in,		
IL_in,		
IL_tl_green,		
IL_out_1,		
IL_out_2,		
ML_in,		
ML_in		
>		

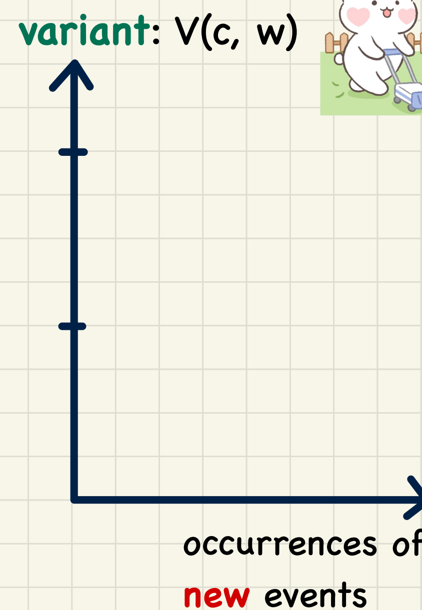
Fixing m2: Measuring Traffic Light Changes

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
    il_pass = 1
  then
    ml_tl := green
    il_tl := red
    ml_pass := 0
  end
```

```
IL_tl_green
  when
    il_tl = red
    b > 0
    a = 0
    ml_pass = 1
  then
    il_tl := green
    ml_tl := red
    il_pass := 0
  end
```

d = 2	ml_pass	il_pass
< init,	1	1
ML_tl_green,	0	1
ML_out_1,	1	1
ML_out_2,	1	1
IL_in,	1	1
IL_in,	1	1
IL_tl_green,	1	0
IL_out_1,	1	1
IL_out_2,	1	1
ML_in,	1	1
ML_in	1	1
>		

variants: *ml_pass* + *il_pass*



PO of Convergence/Non-Divergence/Livelock Freedom

A New Event Occurrence Decreases Variant

$$
\begin{array}{l}
A(c) \\
I(c, v) \\
J(c, v, w) \\
H(c, w) \\
\vdash \\
V(c, F(c, w)) < V(c, w)
\end{array}
\quad \underline{\text{VAR}}
$$

Variants: ml_pass + il_pass

ML_tl_green/VAR

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
    il_pass = 1
  then
    ml_tl := green
    il_tl := red
    ml_pass := 0
  end
```

$$
\begin{array}{lll}
d \in \mathbb{N} & d > 0 & \\
COLOUR = \{green, red\} & green \neq red & \\
n \in \mathbb{N} & n \leq d & \\
a \in \mathbb{N} & b \in \mathbb{N} & c \in \mathbb{N} \\
a + b + c = n & a = 0 \lor c = 0 & \\
ml\_tl \in COLOUR & il\_tl \in COLOUR & \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 & il\_tl = green \Rightarrow b > 0 \land a = 0 & \\
ml\_tl = red \lor il\_tl = red & & \\
ml\_pass \in \{0, 1\} & il\_pass \in \{0, 1\} & \\
ml\_tl = red \Rightarrow ml\_pass = 1 & il\_tl = red \Rightarrow il\_pass = 1 & \\
ml\_tl = red & a + b < d & c = 0 \\
il\_pass = 1 & & \\
\vdash & & \\
0 + il\_pass < ml\_pass + il\_pass & &
\end{array}
$$

PO of Relative Deadlock Freedom

axm0_1	$d \in \mathbb{N}$
axm0_2	$d > 0$
axm2_1	$COLOUR = \{green, red\}$
axm2_2	$green \neq red$
inv0_1	$n \in \mathbb{N}$
inv0_2	$n \leq d$
inv1_1	$a \in \mathbb{N}$
inv1_2	$b \in \mathbb{N}$
inv1_3	$c \in \mathbb{N}$
inv1_4	$a + b + c = n$
inv1_5	$a = 0 \lor c = 0$
inv2_1	$ml\_tl \in COLOUR$
inv2_2	$il\_tl \in COLOUR$
inv2_3	$ml\_tl = green \Rightarrow a + b < d \land c = 0$
inv2_4	$il\_tl = green \Rightarrow b > 0 \land a = 0$
inv2_5	$ml\_tl = red \lor il\_tl = red$
inv2_6	$ml\_pass \in \{0, 1\}$
inv2_7	$il\_pass \in \{0, 1\}$
inv2_8	$ml\_tl = red \Rightarrow ml\_pass = 1$
inv2_9	$il\_tl = red \Rightarrow il\_pass = 1$

Disjunction of ***abstract*** guards

$a + b < d \land c = 0$	**guards of** ML_out **in** m_1
$\lor\ c > 0$	**guards of** ML_in **in** m_1
$\lor\ a > 0$	**guards of** IL_in **in** m_1
$\lor\ b > 0 \land a = 0$	**guards of** IL_out **in** m_1

$\vdash$

Disjunction of ***concrete*** guards

$ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1$	**guards of** ML_tl_green **in** m_2
$\lor\ il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1$	**guards of** IL_tl_green **in** m_2
$\lor\ ml\_tl = green \land a + b + 1 \neq d$	**guards of** ML_out_1 **in** m_2
$\lor\ ml\_tl = green \land a + b + 1 = d$	**guards of** ML_out_2 **in** m_2
$\lor\ il\_tl = green \land b \neq 1$	**guards of** IL_out_1 **in** m_2
$\lor\ il\_tl = green \land b = 1$	**guards of** IL_out_2 **in** m_2
$\lor\ a > 0$	**guards of** ML_in **in** m_2
$\lor\ c > 0$	**guards of** IL_in **in** m_2

Abstract m1

variables: a, b, c

ML_out
when $a + b < d$, $c = 0$
then $a := a + 1$
end

ML_in
when $c > 0$
then $c := c - 1$
end

IL_in
when $a > 0$
then $a := a - 1$, $b := b + 1$
end

IL_out
when $b > 0$, $a = 0$
then $b := b - 1$, $c := c + 1$
end

Concrete m2

ML_tl_green
when $ml\_tl = red$, $a + b < d$, $c = 0$, $il\_pass = 1$
then $ml\_tl := green$, $il\_tl := red$, $ml\_pass := 0$
end

IL_tl_green
when $il\_tl = red$, $b > 0$, $a = 0$, $ml\_pass = 1$
then $il\_tl := green$, $ml\_tl := red$, $il\_pass := 0$
end

ML_out_1
when $ml\_tl = green$, $a + b + 1 \neq d$
then $a := a + 1$, $ml\_pass := 1$
end

IL_out_1
when $il\_tl = green$, $b \neq 1$
then $b := b - 1$, $c := c + 1$, $il\_pass := 1$
end

ML_out_2
when $ml\_tl = green$, $a + b + 1 = d$
then $a := a + 1$, $ml\_tl := red$, $ml\_pass := 1$
end

IL_out_2
when $il\_tl = green$, $b = 1$
then $b := b - 1$, $c := c + 1$, $il\_tl := red$, $il\_pass := 1$
end

IL_in
when $a > 0$
then $a := a - 1$, $b := b + 1$
end

ML_in
when $c > 0$
then $c := c - 1$
end

Discharging POs of m2: Relative Deadlock Freedom

$$
\begin{array}{l}
d \in \mathbb{N} \\
d > 0 \\
COLOUR = \{green, red\} \\
green \neq red \\
n \in \mathbb{N} \\
n \leq d \\
a \in \mathbb{N} \\
b \in \mathbb{N} \\
c \in \mathbb{N} \\
a + b + c = n \\
a = 0 \lor c = 0 \\
ml\_tl \in COLOUR \\
il\_tl \in COLOUR \\
ml\_tl = green \Rightarrow a + b < d \land c = 0 \\
il\_tl = green \Rightarrow b > 0 \land a = 0 \\
ml\_tl = red \lor il\_tl = red \\
ml\_pass \in \{0, 1\} \\
il\_pass \in \{0, 1\} \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\quad\;\; a + b < d \land c = 0 \\
\lor \;\; c > 0 \\
\lor \;\; a > 0 \\
\lor \;\; b > 0 \land a = 0 \\
\vdash \\
\quad\;\; ml\_tl = red \land a + b < d \land c = 0 \land il\_pass = 1 \\
\lor \;\; il\_tl = red \land b > 0 \land a = 0 \land ml\_pass = 1 \\
\lor \;\; ml\_tl = green \\
\lor \;\; il\_tl = green \\
\lor \;\; a > 0 \\
\lor \;\; c > 0
\end{array}
$$

$\vdots$

$$
\begin{array}{|l|}
\hline
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_tl = red \Rightarrow ml\_pass = 1 \\
il\_tl = red \Rightarrow il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1 \\
\hline
\end{array}
\;\ldots\;
\begin{array}{|l|}
\hline
d \in \mathbb{N} \\
d > 0 \\
b \in \mathbb{N} \\
ml\_tl = red \\
il\_tl = red \\
ml\_pass = 1 \\
il\_pass = 1 \\
\vdash \\
\quad\;\; b < d \land ml\_pass = 1 \land il\_pass = 1 \\
\lor \;\; b > 0 \land ml\_pass = 1 \land il\_pass = 1 \\
\hline
\end{array}
\;\ldots\;
\begin{array}{|l|}
\hline
d > 0 \\
b \in \mathbb{N} \\
\vdash \\
b < d \lor b > 0 \\
\hline
\end{array}
\;\textbf{ARI}\;
\begin{array}{|l|}
\hline
d > 0 \\
b > 0 \lor b = 0 \\
\vdash \\
b < d \lor b > 0 \\
\hline
\end{array}
\;\textbf{OR\_L}
\left\{
\begin{array}{l}
\begin{array}{|l|}
\hline
d > 0 \\
b > 0 \\
\vdash \\
b < d \lor b > 0 \\
\hline
\end{array}
\;\textbf{OR\_R2}\;
\begin{array}{|l|}
\hline
d > 0 \\
b > 0 \\
\vdash \\
b > 0 \\
\hline
\end{array}
\;\textbf{HYP} \\[2ex]
\begin{array}{|l|}
\hline
d > 0 \\
b = 0 \\
\vdash \\
b < d \lor b > 0 \\
\hline
\end{array}
\;\textbf{EQ\_LR, MON}\;
\begin{array}{|l|}
\hline
d > 0 \\
\vdash \\
0 < d \lor 0 > 0 \\
\hline
\end{array}
\;\textbf{OR\_R1}\;
\begin{array}{|l|}
\hline
d > 0 \\
\vdash \\
0 < d \\
\hline
\end{array}
\;\textbf{HYP}
\end{array}
\right.
$$

1st Refinement and 2nd Refinement: Provably Correct

Abstract m1

variables: a, b, c

constants: d

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$

invariants:
- **inv1_1** : $a \in \mathbb{N}$
- **inv1_2** : $b \in \mathbb{N}$
- **inv1_3** : $c \in \mathbb{N}$
- **inv1_4** : $a + b + c = n$
- **inv1_5** : $a = 0 \lor c = 0$

variants: $2 \cdot a + b$

```
init
  begin
    a := 0
    b := 0
    c := 0
  end
```

```
ML_out
  when
    a + b < d
    c = 0
  then
    a := a + 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c − 1
  end
```

```
IL_in
  when
    a > 0
  then
    a := a − 1
    b := b + 1
  end
```

```
IL_out
  when
    b > 0
    a = 0
  then
    b := b − 1
    c := c + 1
  end
```

Correctness Criteria:

+ Guard Strengthening
+ Invariant Establishment
+ Invariant Preservation
+ Convergence
+ Relative Deadlock Freedom

Concrete m2

constants: d

sets: $COLOR$

axioms:
- **axm0_1** : $d \in \mathbb{N}$
- **axm0_2** : $d > 0$
- **axm2_1** : $COLOR = \{green, red\}$
- **axm2_2** : $green \neq red$

variables: a, b, c, ml_tl, il_tl, ml_pass, il_pass

invariants:
- **inv2_1** : $ml\_tl \in COLOUR$
- **inv2_2** : $il\_tl \in COLOUR$
- **inv2_3** : $ml\_tl = green \Rightarrow a + b < d \land c = 0$
- **inv2_4** : $il\_tl = green \Rightarrow b > 0 \land a = 0$
- **inv2_5** : $ml\_tl = red \lor il\_tl = red$
- **inv2_6** : $ml\_pass \in \{0, 1\}$
- **inv2_7** : $il\_pass \in \{0, 1\}$
- **inv2_8** : $ml\_tl = red \Rightarrow ml\_pass = 1$
- **inv2_9** : $il\_tl = red \Rightarrow il\_pass = 1$

variants: $ml\_pass + il\_pass$

```
ML_tl_green
  when
    ml_tl = red
    a + b < d
    c = 0
    il_pass = 1
  then
    ml_tl := green
    il_tl := red
    ml_pass := 0
  end
```

```
IL_tl_green
  when
    il_tl = red
    b > 0
    a = 0
    ml_pass = 1
  then
    il_tl := green
    ml_tl := red
    il_pass := 0
  end
```

```
ML_out_1
  when
    ml_tl = green
    a + b + 1 ≠ d
  then
    a := a + 1
    ml_pass := 1
  end
```

```
ML_out_2
  when
    ml_tl = green
    a + b + 1 = d
  then
    a := a + 1
    ml_tl := red
    ml_pass := 1
  end
```

```
IL_out_1
  when
    il_tl = green
    b ≠ 1
  then
    b := b − 1
    c := c + 1
    il_pass := 1
  end
```

```
IL_out_2
  when
    il_tl = green
    b = 1
  then
    b := b − 1
    c := c + 1
    il_tl := red
    il_pass := 1
  end
```

```
ML_in
  when
    c > 0
  then
    c := c − 1
  end
```

```
IL_in
  when
    a > 0
  then
    a := a − 1
    b := b + 1
  end
```